Formelsammlung
Felder und Wellen – WS18/19

1. Ortsvektoren

<table>
<thead>
<tr>
<th>Kartesische Koordinaten</th>
<th>Zylinderkoordinaten</th>
<th>Kugelkoordinaten</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$ =</td>
<td>$R \cdot \cos \varphi$ =</td>
<td>$r \sin \vartheta \cos \varphi$ =</td>
</tr>
<tr>
<td>$y$ =</td>
<td>$R \cdot \sin \varphi$ =</td>
<td>$r \sin \vartheta \sin \varphi$ =</td>
</tr>
<tr>
<td>$z$ =</td>
<td>$z$ =</td>
<td>$r \cos \vartheta$ =</td>
</tr>
<tr>
<td>$\sqrt{x^2 + y^2}$ =</td>
<td>$R$ =</td>
<td>$r \sin \vartheta$ =</td>
</tr>
<tr>
<td>arctan $\frac{y}{x}$ =</td>
<td>$\varphi$ =</td>
<td>$\varphi$ =</td>
</tr>
<tr>
<td>$z$ =</td>
<td>$z$ =</td>
<td>$r \cos \vartheta$ =</td>
</tr>
<tr>
<td>$\sqrt{x^2 + y^2 + z^2}$</td>
<td>$\sqrt{R^2 + z^2}$ =</td>
<td>$r$ =</td>
</tr>
<tr>
<td>arctan $\frac{y}{x}$ =</td>
<td>$\arctan \frac{R}{z}$ =</td>
<td>$\vartheta$ =</td>
</tr>
<tr>
<td>$z$ =</td>
<td>$\varphi$ =</td>
<td>$\varphi$ =</td>
</tr>
</tbody>
</table>

2. Komponenten eines Vektorfeldes

<table>
<thead>
<tr>
<th>$\vec{A}$ = $A_x \vec{e}_x + A_y \vec{e}_y + A_z \vec{e}_z$ =</th>
<th>$A_x \vec{e}_x + A_y \vec{e}_y + A_z \vec{e}_z$ =</th>
<th>$A_x \vec{e}_x + A_y \vec{e}_y + A_z \vec{e}_z$ =</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_x$ = $A_n \cos \varphi - A_z \sin \varphi$ =</td>
<td>$A_n \cos \varphi + A_y \cos \varphi$ =</td>
<td>$A_n \cos \varphi - A_z \sin \varphi$ =</td>
</tr>
<tr>
<td>$A_y$ = $A_n \sin \varphi + A_z \cos \varphi$ =</td>
<td>$A_y$ = $A_n \sin \varphi + A_z \cos \varphi$ =</td>
<td>$A_y$ = $A_n \sin \varphi + A_z \cos \varphi$ =</td>
</tr>
<tr>
<td>$A_z$ = $A_n \cos \varphi - A_y \cos \varphi + A_z \cos \varphi$ =</td>
<td>$A_n \sin \vartheta \cos \varphi + A_y \cos \vartheta \cos \varphi - A_z \sin \varphi$ =</td>
<td>$A_z$ = $A_n \sin \vartheta \cos \varphi - A_y \cos \vartheta \sin \varphi - A_z \sin \varphi$ =</td>
</tr>
<tr>
<td>$-A_x \sin \varphi + A_y \cos \varphi$ =</td>
<td>$A_z$ = $A_n \sin \vartheta \cos \varphi + A_y \cos \vartheta \sin \varphi + A_z \cos \varphi$ =</td>
<td>$A_z$ = $A_n \sin \vartheta \cos \varphi - A_y \cos \vartheta \sin \varphi - A_z \sin \varphi$ =</td>
</tr>
<tr>
<td>$A_y$ = $A_n \sin \varphi$ =</td>
<td>$A_z$ = $A_n \sin \vartheta \cos \varphi - A_y \cos \vartheta \sin \varphi - A_z \sin \varphi$ =</td>
<td>$A_z$ = $A_n \sin \vartheta \cos \varphi - A_y \cos \vartheta \sin \varphi - A_z \sin \varphi$ =</td>
</tr>
<tr>
<td>$A_z \cos \varphi + A_x \sin \varphi$ =</td>
<td>$A_y$ = $A_n \sin \vartheta \cos \varphi + A_y \cos \vartheta \sin \varphi + A_z \cos \varphi$ =</td>
<td>$A_y$ = $A_n \sin \vartheta \cos \varphi + A_y \cos \vartheta \sin \varphi + A_z \cos \varphi$ =</td>
</tr>
<tr>
<td>$A_x \sin \varphi - A_z \cos \varphi$ =</td>
<td>$A_z$ = $A_n \sin \vartheta \cos \varphi - A_y \cos \vartheta \sin \varphi - A_z \sin \varphi$ =</td>
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</tr>
<tr>
<td>$-A_x \sin \varphi + A_y \cos \varphi$ =</td>
<td>$A_z$ = $A_n \sin \vartheta \cos \varphi - A_y \cos \vartheta \sin \varphi - A_z \sin \varphi$ =</td>
<td>$A_z$ = $A_n \sin \vartheta \cos \varphi - A_y \cos \vartheta \sin \varphi - A_z \sin \varphi$ =</td>
</tr>
</tbody>
</table>
3. Linien-, Flächen- und Volumenelemente

Kartesische Koordinaten
\[ ds = e_x \, dx + e_y \, dy + e_z \, dz \]
\[ d\vec{r} = e_x \cdot dy \, dz + e_y \cdot dx \, dz + e_z \cdot dx \, dy \]
\[ dv = dx \, dy \, dz \]

Zylinderkoordinaten
\[ ds = e_r \cdot dR + e_z \cdot Rd\varphi + e_z \cdot dz \]
\[ d\vec{r} = e_r \cdot R \cdot d\varphi \, dz + e_z \cdot dR \cdot d\varphi \]
\[ dv = R \, dR \, d\varphi \, dz \]

Kugelkoordinaten
\[ ds = e_r \cdot dr + e_\vartheta \cdot r \, d\vartheta \]
\[ d\vec{r} = e_r \cdot r^2 \cdot \sin \vartheta \, d\vartheta \, d\varphi + e_\vartheta \cdot r \cdot \sin \vartheta \, dr \, d\varphi + e_\varphi \cdot r \, dr \, d\vartheta \]
\[ dv = r^2 \cdot \sin \vartheta \cdot dr \, d\vartheta \, d\varphi \]

4. Differentialoperatoren

Kartesische Koordinaten
\[ \text{grad } \psi = e_x \frac{\partial \psi}{\partial x} + e_y \frac{\partial \psi}{\partial y} + e_z \frac{\partial \psi}{\partial z} \]
\[ \text{div } \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \]
\[ \text{rot } \vec{A} = e_x \left( \frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y} \right) + e_y \left( \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) + e_z \left( \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right) \]
\[ \Delta \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \]

Zylinderkoordinaten
\[ \text{grad } \psi = e_r \frac{\partial \psi}{\partial r} + e_\varphi \frac{1}{r} \frac{\partial \psi}{\partial \varphi} + e_z \frac{\partial \psi}{\partial z} \]
\[ \text{div } \vec{A} = \frac{1}{r} \frac{\partial}{\partial r} \left( r A_r \right) + \frac{1}{r \sin \vartheta} \frac{\partial}{\partial \vartheta} \left( A_\vartheta \sin \vartheta \right) \]
\[ + \frac{1}{r \sin \vartheta} \frac{\partial A_\varphi}{\partial \varphi} \]
\[ \text{rot } \vec{A} = e_r \left( \frac{\partial A_\varphi}{\partial z} - \frac{\partial A_z}{\partial \varphi} \right) + e_\varphi \left( \frac{\partial A_z}{\partial r} - \frac{\partial A_r}{\partial z} \right) + e_z \left( \frac{\partial A_r}{\partial \varphi} - \frac{\partial A_\varphi}{\partial r} \right) \]
\[ \Delta \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \vartheta} \frac{\partial}{\partial \vartheta} \left( \sin \vartheta \frac{\partial \psi}{\partial \vartheta} \right) \]
\[ + \frac{1}{r^2 \sin^2 \vartheta} \frac{\partial^2 \psi}{\partial \varphi^2} \]

Kugelkoordinaten
\[ \text{grad } \psi = e_\varrho \frac{\partial \psi}{\partial \varrho} + e_\vartheta \frac{1}{\varrho} \sin \vartheta \frac{\partial \psi}{\partial \vartheta} + e_\varphi \frac{1}{\varrho \sin \vartheta} \frac{\partial \psi}{\partial \varphi} \]
\[ \text{div } \vec{A} = \frac{1}{\varrho} \frac{\partial}{\partial \varrho} \left( \varrho A_\varrho \right) + \frac{1}{\varrho \sin \vartheta} \frac{\partial}{\partial \vartheta} \left( A_\vartheta \sin \vartheta \right) \]
\[ + \frac{1}{\varrho \sin \vartheta} \frac{\partial A_\varphi}{\partial \varphi} \]
\[ \text{rot } \vec{A} = e_\varrho \left( \frac{\partial A_\vartheta}{\partial \varphi} - \frac{\partial A_\varphi}{\partial \vartheta} \right) + e_\vartheta \left( \frac{\partial A_\varphi}{\partial \varrho} - \frac{\partial A_\varrho}{\partial \varphi} \right) + e_\varphi \left( \frac{\partial A_\varrho}{\partial \vartheta} - \frac{\partial A_\vartheta}{\partial \varrho} \right) \]
\[ \Delta \psi = \frac{1}{\varrho^2} \frac{\partial}{\partial \varrho} \left( \varrho^2 \frac{\partial \psi}{\partial \varrho} \right) + \frac{1}{\varrho \sin \vartheta} \frac{\partial}{\partial \vartheta} \left( \sin \vartheta \frac{\partial \psi}{\partial \vartheta} \right) \]
\[ + \frac{1}{\varrho^2 \sin^2 \vartheta} \frac{\partial^2 \psi}{\partial \varphi^2} \]
5. Maxwellgleichungen in allgemeingültiger Form

\begin{align*}
\text{div } \vec{D} &= \rho \\
\text{rot } \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t} \\
\text{div } \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\text{rot } \vec{B} &= 0
\end{align*}

\[ \int \vec{D} \cdot d\vec{f} = \int \rho \, dv \]
\[ \int \vec{H} \cdot d\vec{s} = \int \vec{J} \cdot d\vec{f} + \int \frac{\partial \vec{D}}{\partial t} \cdot d\vec{f} \]

6. Materialgleichungen

\[ \vec{D} = \varepsilon_0 \vec{E} + \vec{P} \]
\[ \vec{B} = \mu_0 \left( \vec{H} + \vec{M} \right) \]
\[ \text{allgemein :} \]
\[ \varepsilon_0 \varepsilon_r \vec{E} \]
\[ \mu_0 \mu_r \vec{H} \]
\[ \text{für lineare, isotrope Medien:} \]
\[ \vec{P} = \chi_d \varepsilon_0 \vec{E} \]
\[ \vec{M} = \chi_m \vec{H} \]
\[ \text{mit } \chi_d = \varepsilon_r - 1 \]
\[ \text{mit } \chi_m = \mu_r - 1 \]

7. Kräfte und Momente

\[ \vec{F} = \vec{Q} \cdot \vec{E} \]
\[ \text{Kraft zwischen zwei Ladungen} \]
\[ \vec{F} = \frac{1}{4\pi \varepsilon} \frac{Q_1 Q_2}{r^2} \hat{e}_r \]
\[ \vec{F} = \vec{Q} \cdot (\vec{v} \times \vec{B}) \]
\[ = \ell \cdot \vec{A} \cdot (\vec{j} \times \vec{B}) \]
\[ = I \left( \ell \times \vec{B} \right) \]

8. Grenzflächen

\[ \sigma = D_{n_2} - D_{n_1} \]
\[ E_{t_2} = E_{t_1} \]
\[ J_f = H_{t_2} - H_{t_1}, J_f \perp \left( H_{t_2} - H_{t_1} \right) \]
\[ B_{n_2} = B_{n_1} \]

9. Feldenergiedichte

\[ w_e = \frac{1}{2} \vec{E} \cdot \vec{D} \]
\[ w_m = \frac{1}{2} \vec{H} \cdot \vec{B} \]
\[ \text{allgemein :} \]
\[ w_e = \frac{1}{2} \varepsilon \vec{E}^2 \]
\[ w_m = \frac{1}{2} \mu \vec{H}^2 \]
\[ \text{für lineare, isotrope Medien:} \]
\[ \text{Gesamtenergie} \]
\[ W_e = \int w_e \, dv \]
\[ = \frac{1}{2} \int \Phi \rho \, dv \]
\[ W_m = \int w_m \, dv \]
10. Skalarpotential

Elektrostatik: \[ \Phi_{\text{el}} (\mathbf{r}) = \Phi_{\text{el}} (\mathbf{r}) = - \int_{\mathbf{r}} \mathbf{E} \ d\mathbf{s} \quad \mathbf{E} = - \nabla \Phi_{\text{el}} \]

Coulomb integral: \[ \Phi (\mathbf{r}) = \frac{1}{4\pi \varepsilon} \int \frac{\rho (\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \ dv' \]

Poissongleichung: \[ \Delta \Phi_{\text{el}} = - \frac{\rho}{\varepsilon} \]

Laplacegleichung: \[ \Delta \Phi_{\text{el}} = 0 \quad (\text{für} \quad \rho = 0) \]

Partikulärlosung in kartesischen Koordinaten:
(für \( \alpha \neq 0 \)) \[ \Phi_{\text{el}} = \begin{bmatrix} a_1 \sin (\alpha x) + a_2 \cos (\alpha x) \cdot \left[ a_3 \sin (\beta y) + a_4 \cos (\beta y) \right] \
\cdot a_5 \cdot e^{-\gamma z} + a_6 \cdot e^{\gamma z} \end{bmatrix} \quad \text{mit} \quad \gamma^2 = \alpha^2 + \beta^2 \]

11. Vektorpotential

Vektorpotential: \[ \text{rot} \ \vec{A} = \vec{B} \]

Coulomb-Eichung: \[ \text{div} \ \vec{A} = 0 \]

"Poissongleichung": \[ \Delta A_x = - \mu \cdot J_x \quad \Delta A_y = - \mu \cdot J_y \quad \Delta A_z = - \mu \cdot J_z \]

"Coulomb integral": \[ \vec{A} (\mathbf{r}) = \frac{\mu}{4\pi} \int \frac{\mathbf{J} (\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \ dv' \]

Gesetz v. Biot-Savart: \[ \vec{B} (\mathbf{r}) = \frac{\mu}{4\pi} \int \frac{\mathbf{J} (\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \ dv' \]

für Linienleiter: \[ \vec{A} (\mathbf{r}) = \frac{\mu I}{4\pi} \int \frac{d\mathbf{s}'}{|\mathbf{r} - \mathbf{r}'|} \quad \vec{B} (\mathbf{r}) = \frac{\mu I}{4\pi} \int \frac{d\mathbf{s} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \]

12. Das stationäre Strömungsfeld

Stromdichte: \[ \mathbf{J} = \kappa \ \mathbf{E} \quad \mathbf{J} = \rho_+ \mathbf{v}_+ + \rho_- \mathbf{v}_- \]

elektrischer Widerstand \[ R = \frac{U}{I} = \frac{-\int \mathbf{E} \ d\mathbf{s}}{\int \mathbf{J} \ d\mathbf{r}} \]

Verlustleistungsichte: \[ \frac{d\mathcal{L}}{dt} = \mathbf{J} \cdot \mathbf{E} \]
13. Kapazität

allgemein: \( Q_i = \sum_{k=1}^{N} c_{ik} \Phi_k \)

\( \Phi_i = \sum_{k=1}^{N} p_{ik} Q_k \) Influenzkoeffizient

\( \Phi_i = \sum_{k=1}^{N} p_{ik} Q_k \) Potentialkoeffizient

speziell:

für \( n = 2 \) und \( Q_1 = -Q_2 = Q \): \( C = \frac{Q}{U} = \frac{\oint \vec{D} \cdot d\vec{r}}{-\int \vec{E} \cdot dS} \)

Energie allgemein: \( W_e = \frac{1}{2} \sum_{i=1}^{N} \sum_{k=1}^{N} p_{ik} Q_i Q_k \)

Energie speziell: für \( n = 2 \) und \( Q_1 = -Q_2 = Q \): \( W_{el} = \frac{1}{2} C \cdot U^2 \)

14. Induktivität

Energie:

\( W_m = \frac{1}{2} \sum_{i=1}^{N} \sum_{k=1}^{N} L_{ik} I_i I_k \)

für \( n = 1 \): \( W_m = \frac{1}{2} L \cdot I^2 \)

für \( n = 2 \): \( W_m = \frac{1}{2} L_{11} I_1^2 + L_{12} I_1 I_2 + \frac{1}{2} L_{22} I_2^2 \)

Gegeninduktivität: \( L_{ik} = \frac{N_k \cdot \Phi_{m,ik}}{I_i} \)

äußere Selbstinduktivität: \( L^{(a)} = \frac{N \cdot \Phi^{(a)}_m}{I} \)

Magnetischer Fluss: \( \Phi_{m,ik} = \int \oint_{F_i} \vec{B} \cdot d\vec{r} \)

Induktionsspannung: \( U_{ind,ik} = -N_k \frac{d\Phi_{m,ik}}{dt} \)

15. Maxwellgleichungen für harmonische Vorgänge

\( \text{div} \ \vec{D} = \rho \)

\( \text{rot} \ \vec{H} = \vec{J} + j \omega \vec{D} \)

\( \text{rot} \ \vec{E} = -j \omega \vec{B} \)

\( \text{div} \ \vec{B} = 0 \)
16. Schnell veränderliche Felder

Allgemein:

\[ \Delta \vec{E} - \varepsilon \mu \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \quad \text{leitende Materialien} \]
\[ \Delta \vec{E} - \kappa \mu \frac{\partial \vec{E}}{\partial t} - \varepsilon \mu \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \quad \text{nicht leitende Materialien} \]

Harmonische Vorgänge

\[ \Delta \vec{E} + \omega^2 \varepsilon \mu \vec{E} = 0 \quad \text{leitende Materialien} \]
\[ \Delta \vec{E} - j \omega \kappa \mu \vec{E} + \omega^2 \varepsilon \mu \vec{E} = 0 \quad \text{nicht leitende Materialien} \]

ebene Wellen:

\[ \vec{E} = E_0 e^{j(\omega t + kx)} \hat{e}_y \]
\[ \vec{H} = H_0 e^{j(\omega t + kx)} \hat{e}_z \]
\[ H_0 = \mp \frac{\varepsilon}{\sqrt{\mu}} E_0 \]

Wellenzahl:

\[ k^2 = \omega^2 \varepsilon \mu = \left(\frac{2\pi}{\lambda}\right)^2 \]

Phasengeschwindigkeit:

\[ c = \frac{1}{\sqrt{\varepsilon \mu}} \]

Komplexer Poyntingvektor

\[ \vec{S} = \frac{1}{2} \vec{E} \times \vec{H}^* \]

Zeitlicher Mittelwert

\[ \vec{S}_{av} = \frac{1}{2} \text{Re}\{\vec{E} \times \vec{H}^*\} \]
17. Verwendete Formelzeichen

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
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<td>$d\tilde{s}$, $d\tilde{r}$, $dv$</td>
<td>Weg-, Flächen- und Volumenelemente</td>
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<tr>
<td>$\tilde{E}, \tilde{D}$</td>
<td>elektrische Feldstärke, Verschiebungsdichte</td>
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<tr>
<td>$\tilde{H}, \tilde{B}$</td>
<td>magnetische Feldstärke, Flußdichte</td>
</tr>
<tr>
<td>$\tilde{J}, \tilde{J}_f$</td>
<td>Stromdichte, Flächenstromdichte</td>
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<tr>
<td>$\tilde{P}$</td>
<td>Polarisation</td>
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<td>$\tilde{M}$</td>
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<td>$\varepsilon_0$, $\varepsilon_r$</td>
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<tr>
<td>$\mu_0$, $\mu_r$</td>
<td>Permeabilitätskonstante, -zahl</td>
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<tr>
<td>$\chi_{el}$, $\chi_m$</td>
<td>elektrische, magnetische Suszeptibilität</td>
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<tr>
<td>$Q$</td>
<td>Ladung</td>
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<tr>
<td>$\ell$</td>
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<td>Windungszahl</td>
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<tr>
<td>$U$, $I$</td>
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<td>$R$</td>
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<td>$C$, $L$</td>
<td>Kapazität, Induktivität</td>
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<td>$c_{ik}$, $p_{ik}$</td>
<td>Influenzkoeffizienten, Potentialkoeffizienten</td>
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<td>$L_{ik}$</td>
<td>Induktionskoeffizienten,</td>
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<td>$\vec{n}$</td>
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<tr>
<td>$w$, $W$</td>
<td>Energiedichte, Energie</td>
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<td>$c$, $\lambda$</td>
<td>Lichtgeschwindigkeit, Wellenlänge</td>
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<tr>
<td>$\tilde{S}$</td>
<td>komplexer Poyntingvektor</td>
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<tr>
<td>$\tilde{S}_{av}$</td>
<td>Zeitlicher Mittelwert der Energiestromdichte</td>
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